

# Maximal independent sets in Borel graphs and large cardinals

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## Abstract

We construct a Borel graph  $G$  such that  $ZF + DC +$  "There are no maximal independent sets in  $G$ " is equiconsistent with  $ZFC +$  "There exists an inaccessible cardinal".<sup>1</sup>

## Introduction

The main result of this note is motivated by our recent study of maximal almost disjoint families and their relatives. Recall that  $\mathcal{F} \subseteq [\omega]^\omega$  is a MAD family if  $A \neq B \in \mathcal{F} \rightarrow |A \cap B| < \aleph_0$ , and  $\mathcal{F}$  is maximal with respect to this property. Maximal eventually different (MED) families are the analog of MAD families where elements of  $[\omega]^\omega$  are replaced by graphs of functions from  $\omega$  to  $\omega$ , namely,  $f, g \in \omega^\omega$  are eventually different if  $f(n) \neq g(n)$  for large enough  $n$ , and  $\mathcal{F} \subseteq \omega^\omega$  is a MED family if the elements of  $\mathcal{F}$  are pairwise eventually different and  $\mathcal{F}$  is maximal with respect to this property.

Questions on the (non-)existence and definability of such families have attracted considerable interest for decades. The first results were obtained by Mathias who proved the following theorem:

**Theorem [Ma]:** There are no analytic MAD families.

As for the possibility of the non-existence of MAD families, the following result was recently proved by the authors (earlier such results were proven by Mathias in [Ma] and by Toernquist in [To] using Mahlo and inaccessible cardinals, respectively):

**Theorem [HwSh:1090]:**  $ZF + DC +$  "There are no MAD families" is equiconsistent with  $ZFC$ .

Quite surprisingly, the situation for MED families turns out to be different:

**Theorem [HwSh:1089]:** Assuming  $ZF$ , there exists a Borel MED family.

A possible approach to explaining the above difference is via Borel combinatorics. The study of Borel and analytic graphs was initiated by Kechris, Solecki and Todorcevic in [KST], and has been a source of fruitful research ever since (see [KM] for a survey of recent results). The above questions on MAD families are connected

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to Borel combinatorics due to the following observation: There exist Borel graphs  $G_{MAD}$  and  $H_{MED}$  such that there exists a MAD (MED) family iff there exists a maximal independent set in  $G_{MAD}$  ( $H_{MED}$ ). Therefore, we might try to explain the above difference of MAD and MED families by pursuing the general problem of classifying Borel graphs according to the consistency strength of  $ZF + DC +$  "There are no maximal independent sets in  $G$ ".

The main goal of this note is to show that for some Borel graphs  $G$ ,  $ZF + DC +$  "There are no maximal independent sets in  $G$ " has large cardinal strength.

## The main result

**Definition 1:** We shall define a Borel graph  $G = (V, E)$  as follows:

- a.  $V$  is the set of reals  $r$  that code the following objects:
  - 1. A linear order  $I_r$  of the element of  $\omega$  or some  $n < \omega$ .
  - 2. A sequence  $(s_{r,\alpha} : \alpha \in I_r)$  of pairwise distinct reals.
  - 3. A sequence of functions  $(f_{r,a} : a \in I_r)$  such that each  $f_{r,a}$  is an injective function from  $I_{r,<a} := \{b \in I_r : b <_{I_r} a\}$  onto some initial segment of  $\omega$ .
- b. Given  $r_1 \neq r_2 \in V$  and  $b \in I_{r_2}$ , let  $X_{r_1,r_2,b}$  be the set of pairs  $(a_1, a_2) \in I_{r_1} \times I_{r_2,<b}$  such that  $s_{r_1,a_1} = s_{r_2,a_2}$ .
- c. Given  $r_1 \neq r_2 \in V$ ,  $\neg(r_1 E r_2)$  holds iff one of the following holds:
  - 1. There exists  $b \in I_{r_2}$  such that  $X_{r_1,r_2,b}$  is an isomorphism from  $I_{r_1}$  to  $I_{r_2,<b}$  which also commutes with  $f_{-, -}$ .
  - 2. There exists  $b \in I_{r_1}$  such that  $X_{r_2,r_1,b}$  is an isomorphism from  $I_{r_2}$  to  $I_{r_1,<b}$  which also commutes with  $f_{-, -}$ .

**Definition 2:** Given  $r_1 \neq r_2 \in V$ , we say that  $r_2$  extends  $r_1$  and denote it by  $r_1 <_G r_2$  when  $\neg(r_1 E r_2)$  and clause (1) holds in definition 1(c).

**Claim 3** ( $ZF + DC$ ): Let  $X \subseteq V$  be an independent set.

- a.  $X$  is linearly ordered by  $<_G$ .
- b. If  $X$  is countable then  $X$  is not a maximal independent set.

**Proof:** a. Obvious.

b. By clause (a), there is a linear order  $I$  such that  $X = \{r_i : i \in I\}$  and  $i <_I j$  iff  $r_i <_G r_j$ . For every  $i < j \in I$ , let  $F_{i,j}$  be the isomorphism from  $I_{r_i}$  to a proper initial segment of  $I_{r_j}$  witnessing  $r_i <_G r_j$ . Let  $I_r$  be the direct limit of the system  $(I_{r_i}, F_{j,k} : i, j, k \in I, j < k)$ . For  $a \in I_r$ , let  $s_{r,a}$  be  $s_{r_i,a'}$  where  $a' \in I_{r_i}$  is some representative of  $a$ , and define  $f_{r,a}$  similarly. Let  $r \in V$  be a real coding  $I_r$ ,  $(s_{r,a} : a \in I_r)$  and  $(f_{r,a} : a \in I_r)$ , then  $\neg(r E r_i)$  for every  $r_i \in X$ .  $\square$

**Theorem 4:**  $ZF + DC +$  "There is no maximal independent set in  $G$ " is equiconsistent with  $ZFC +$  "There exists an inaccessible cardinal".

Theorem 4 will follow from the following claims:

**Claim 5** ( $ZF + DC$ ): If there exists  $a \in \omega^\omega$  such that  $\aleph_1 = \aleph_1^{L[a]}$ , then there exists a maximal independent set in  $G$ .

**Claim 6:** There is no maximal independent set in  $G$  in Levy's model (aka Solovay's model).

Remark: While the set of vertices of  $G$  is denoted by  $V$ , the set-theoretic universe will be denoted by  $\mathbf{V}$ .

**Proof of claim 5:** Let  $(s_\alpha : \alpha < \omega_1^{L[a]}) \in L[a]$  be a sequence of pairwise distinct reals, and let  $\bar{f}^* = (f_\alpha^* : \alpha < \omega_1^{L[a]}) \in L[a]$  be a sequence of functions such that each  $f_\alpha^*$  is an injective function from  $\alpha$  onto  $|\alpha| \leq \omega$ . For each  $\alpha < \omega_1^{L[a]}$ , let  $r_\alpha \in (\omega^\omega)^{L[a]}$  be the  $<_{L[a]}$ -first real that codes  $(\alpha, (s_\beta : \beta < \alpha), \bar{f}^* \upharpoonright \alpha)$ . The sequences  $(s_\alpha : \alpha < \omega_1^{L[a]})$ ,  $(f_\alpha^* : \alpha < \omega_1^{L[a]})$  and  $(r_\alpha : \alpha < \omega_1^{L[a]})$  belong to  $\mathbf{V}$ , and as  $\omega_1 = \omega_1^{L[a]}$ , their length is  $\omega_1$ .

It's easy to see that  $\{r_\alpha : \alpha < \omega_1^{L[a]}\}$  is a well-defined set and is an independent subset of  $V$ , we shall prove that it's a maximal independent set. Let  $r \in V \setminus \{r_\alpha : \alpha < \omega_1^{L[a]}\}$  and suppose towards contradiction that  $\neg(rEr_\alpha)$  for every  $\alpha < \omega_1^{L[a]}$ . There are two possible cases:

**Case I:**  $r_\alpha <_G r$  for every  $\alpha < \omega_1^{L[a]}$ . In this case,  $I_r$  is a linear order, and each  $\alpha < \omega_1^{L[a]}$  embeds into  $I_r$  as an initial segment. Therefore,  $\omega_1 = \omega_1^{L[a]}$  embeds into  $I_r$  as an initial segment, a contradiction.

**Case II:**  $r <_G r_\alpha$  for some  $\alpha < \omega_1^{L[a]}$ . Let  $\alpha$  be the minimal ordinal with this property, then  $\alpha$  necessarily has the form  $\beta+1$ . If  $r = r_\beta$ , then we get a contradiction to the choice of  $r$ . If  $r \neq r_\beta$ , then it's easy to see that  $rEr_\beta$ , contradicting our assumption.  $\square$

**Proof of claim 6:** Let  $\kappa$  be an inaccessible cardinal and let  $\mathbb{P} = \text{Coll}(\aleph_0, < \kappa)$ , we shall prove that  $\Vdash_{\mathbb{P}}$  "There is no maximal independent set in  $G$  from  $HOD(\mathbb{R})$ ". Suppose towards contradiction that  $p \in \mathbb{P}$  forces that  $\tilde{X}$  is such a set. Let  $\mathbb{Q}$  be a forcing notion such that  $\mathbb{Q} \triangleleft \mathbb{P}$ ,  $|\mathbb{Q}| < \kappa$ ,  $p \in \mathbb{Q}$  and  $\tilde{X}$  is definable using a parameter from  $\mathbb{R}^{\mathbf{V}^{\mathbb{Q}}}$ . By the properties of the Levy collapse, we may assume wlog that  $\mathbb{Q} = \{0\}$  and  $p = 0$ . If  $\Vdash_{\mathbb{P}} \tilde{X} \subseteq (\omega^\omega)^{\mathbf{V}}$ , then  $\Vdash_{\mathbb{P}} |\tilde{X}| = \aleph_0$ , and by claim 3,  $\tilde{X}$  is not a maximal independent set in  $\mathbf{V}^{\mathbb{P}}$ , a contradiction. Therefore, there exist  $p_1 \in \mathbb{P}$  and  $\tilde{r}_1$  such that  $p_1 \Vdash_{\mathbb{P}} \tilde{r}_1 \in \tilde{X} \wedge \tilde{r}_1 \notin \mathbf{V}$ . Let  $\mathbb{Q}_1 \triangleleft \mathbb{P}$  be a forcing of cardinality  $< \kappa$  such that  $p_1 \in \mathbb{Q}_1$  and  $\tilde{r}_1$  is a  $\mathbb{Q}_1$ -name. For  $l = 2, 3$  let  $(\mathbb{Q}_l, p_l, \tilde{r}_l)$  be isomorphic copies of  $(\mathbb{Q}_1, p_1, \tilde{r}_1)$  such that  $\prod_{n=1,2,3} \mathbb{Q}_n \triangleleft \mathbb{P}$  (identifying  $\mathbb{Q}_1$  with its canonical image in the product). Choose  $(p_1, p_2) \leq (q_1, q_2)$  such that  $(q_1, q_2) \Vdash_{\mathbb{Q}_1 \times \mathbb{Q}_2} \tilde{r}_1 \neq \tilde{r}_2$ . As  $(q_1, q_2) \Vdash_{\mathbb{Q}_1 \times \mathbb{Q}_2} \tilde{r}_1, \tilde{r}_2 \in \tilde{X}$ , then wlog  $(q_1, q_2)$  forces that  $\tilde{r}_1 <_G \tilde{r}_2$  as witnessed by an isomorphism from  $I_{\tilde{r}_1}$  to  $I_{\tilde{r}_2, < s}$  for some  $s \in I_{\tilde{r}_2}$ . Let  $q_3 \in \mathbb{Q}_3$  be the conjugate of

$q_1$ , then  $(q_2, q_3)$  forces (in  $\mathbb{Q}_2 \times \mathbb{Q}_3$ ) that  $r_2, r_3 \in \tilde{X}$  and  $r_3 <_G r_2$  as witnessed by an isomorphism from  $I_{r_3}$  to  $I_{r_2, <_s}$ . Now pick  $(q_1, q_2, q_3) \leq (q'_1, q'_2, q'_3)$  that forces in addition that  $r_1 \neq r_3$ , then necessarily it forces that  $r_1 E r_3$ , a contradiction.  $\square$

## Open problems

**Notation:** Given a Borel graph  $G$ , let  $\psi(G)$  be the statement "There are no maximal independent sets in  $G$ ".

**Problem 1:** Classify the Borel graphs according to the consistency strength of  $ZF + DC + \psi(G)$ .

As the above problem seems to be quite difficult at the moment, it might be reasonable to consider the following subproblems first:

**Problem 2:** What are the possibilities (in terms of large cardinal strength) for the consistency strength of  $ZF + DC + \psi(G)$ ?

**Problem 3:** Find combinatorial/descriptive set theoretic/model theoretic properties  $\phi_1$  and  $\phi_2$  such that:

- a.  $\phi_1(G_{MAD})$ .
- b.  $\phi_2(H_{MED})$ .
- c.  $\phi_1(G) \rightarrow ZF + DC + \psi(G)$  is equiconsistent with  $ZFC$ .
- d.  $\phi_2(G) \rightarrow ZF + DC \vdash \neg\psi(G)$ .
- e.  $\phi_1$  and  $\phi_2$  are satisfied by a large collection of Borel graphs.

A solution to problem (3) would explain the difference between MAD and MED families that was discussed in the introduction.

## References

- [HwSh:1089] Haim Horowitz and Saharon Shelah, A Borel maximal eventually different family, arXiv:1605.07123.
- [HwSh:1090] Haim Horowitz and Saharon Shelah, Can you take Toernquist's inaccessible away?, arXiv:1605.02419.
- [KM] A. S. Kechris and A. Marks, Descriptive graph combinatorics, Manuscript, 2015.
- [KST] A. S. Kechris, S. Solecki and S. Todorcevic, Borel chromatic numbers, Adv. Math. **141** (1999), 1-44.
- [Ma] A. R. D Mathias, Happy families, Ann. Math. Logic **12** (1977), no. 1, 59-111. MR 0491197.
- [To] Asger Toernquist, Definability and almost disjoint families, arXiv:1503.07577.

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